28) To derive the optimal iteration formulas for polynomials of degree ”d”

Not being a mathematician, I’ll talk little about the origin about the magic iteration formulas for quadratics, cubics, and quartics. Generally there is no way of parameter setups for a polynomial that can be claimed as right or wrong.

Background: According to the great mathematicians any polynomials of degree ”d” (d = 2, 3, 4, etc) can be rewritten to a form with d - 1 parameters, with the parameter ”1” before the term z^d, and ”0” before the term z^{d-1}. They call such a polynomial monic and centered. That's why there usually is no ”z” in quadratics, no ”z^2” in cubics, and no ”z^3” in quartics. When drawing pictures of the parameter space, the variable ”z” has to be initiated to a critical point, which is obtained by putting the derivative to zero. For quadratics we get:

$p(z) = z^2 + c$
$p'(z) = 2z = 0$
$z = 0$

For cubics, which happens to have two critical points, we would get:

$p(z) = z^3 + Az + b$
$p'(z) = 3z^2 + A = 0$
$z^2 = - A/3$
z = +sqrt(- A/3), and z = - sqrt(- A/3)

This variant of cubics is availible in my ik3-module as ”DevaneyIICubic3”. To get more ”easy to handle” critical points for cubics there's reason to rewrite the parameter ”A” before ”z” to $-3a^2$. Thus:

$p(z) = z^3 - 3a^2z + b$
$p'(z) = 3z^2 - 3a^2 = 0$
$3z^2 = 3a^2$
$z^2 = a^2$
z = +a and z = -a.

For quartics, third degree equations have to be solved. That’s not the most easy task (ask Stig). Now I happen to have a copy of an article of the great mathematicians Bodil Branner and John Hamal Hubbard The iteration of cubic polynomials. *Part I: The global topology of parameter space* (Acta Mathematica 160, p143 - 205). I am a little bit ashamed for that, because I hardly understand a bit of it, since most of the article is written in an obscure ”algebraic topology language”. However I send it to Stig, because on p150 - 151 there seems to be a general formula for deriving the ”optimal” iteration formula for a polynomial of any degree. What we found out together was the following:

Deriving the optimal formulas for iteration: The main idea is to go the opposite way. Instead of deriving the critical points from a polynomial, the great fellows start with the critical points and from these, they construct the parameters. Then there is no need for solving obscure equations. The general magic formula then runs as:

$p'(z) = d(z - a_1)(z - a_2)\ldots\ldots\ldots\ldots(z - a_{d-1})$

$a_1 + a_2 + \ldots\ldots\ldots\ldots + a_{d-1} = 0$.
The terms a_1 , a_2, …………… a_{d-1} are the critical points.

1) Case d = 2, quadratic polynomials.
$p'(z) = 2(z - a_1) = 2z - 2a_1$
$a_1 = 0$ gives
$p'(z) = 2z$
Now we integrate:
$p(z) = z^2 + b$
This gives the well known iteration formula

$$\mathbf{z \to z^2 + c}$$

the Mandelbrot set constituted of those parameters ”c” for which the critical point z = 0 has a bounded orbit (see figure 1, the Mandelbrot set).

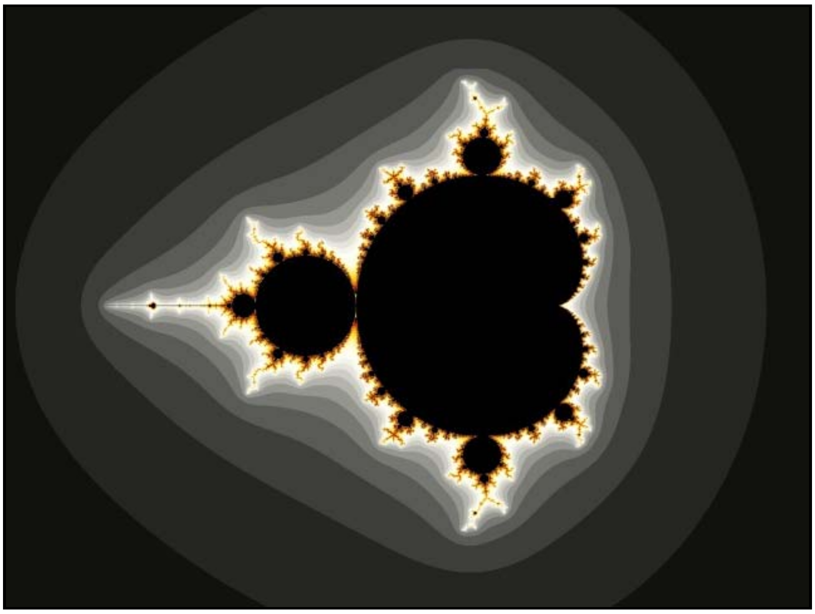

Fig 1. The Mandelbrot set.

2) Case d = 3, cubic polynomials.
$p'(z) = 3(z - a_1)(z - a_2) = 3(z^2 - a_2 z - a_1 z + a_1 a_2)$
$a_1 + a_2 = 0$ gives us $a_2 = - a_1$ and
$p'(z) = 3(z^2 - a_1^2) = 3z^2 - 3a_1^2$
Now we integrate:
$p(z) = z^3 - 3a_1^2 z + b$
This gives the now well known iteration formula for cubics,

$$\mathbf{z \to z^3 - 3a^2 z + b}$$

critical points z = +a and z = -a. The set M+ are those parameters (a, b) for which z = +a has a bounded orbit, and the set M- are those parameters (a, b) for which z = -a has a bounded orbit. The full cubic analogy to the Mandelbrot set are those parameters (a, b) for which both z = +a and z = -a have bounded orbits, that is the set which is common to M+ and M-. As we have two complex parameters, ”a” and ”b” the cubic parameter space is a four dimensional hyper space. The special slice of the full cubic analogy when "a" is fixed to zero, that is we iterate $z \to z^3 + b$, is the well known Cubic

Mandelbrot set (see figure 2, the Generalized Mandelbrot set for Cubics). Further more can be read at my "Cubic Tutorial":

http://klippan.seths.se/ik/CubTut/cubictut.html

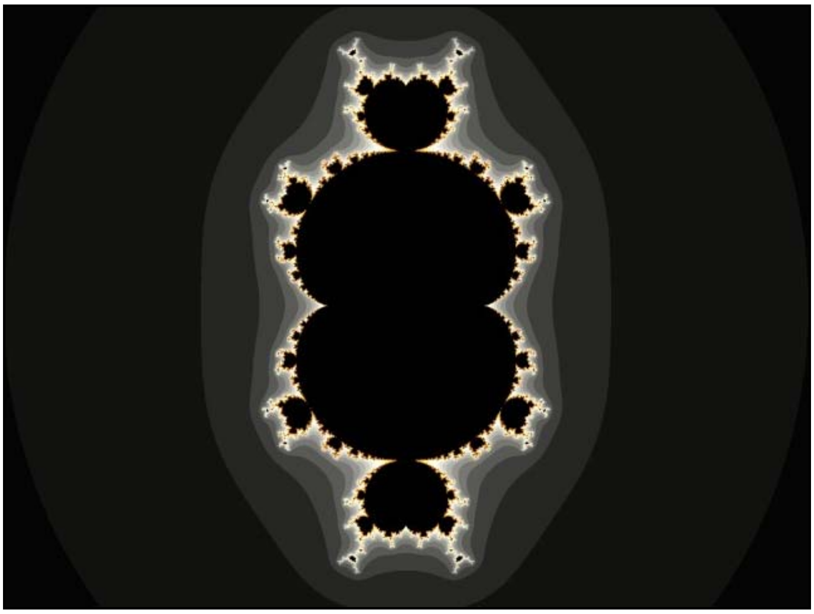

Fig 2. The Generalized Mandelbrot set for Cubics.

3) Case d = 4, quartic polynomials.

$p'(z) = 4(z - a_1)(z - a_2)(z - a_3)$.
For convenience, set $a_1 = a$, $a_2 = b$, $a_3 = c$.
$p'(z) = 4(z - a)(z - b)(z - c) = 4(z^2 - bz - az + ab)(z - c) =$
$4(z^3 - az^2 - bz^2 + abz - cz^2 + acz + bcz - abc)$.

$a + b + c = 0$ gives $c = - (a + b)$, and $p'(z) = 4(z^3 + abz - a(a + b)z - b(a + b)z + ab(a + b)) = 4z^3 + 4abz - 4 a(a + b)z - 4b(a + b)z + 4ab(a + b)$.

Now we integrate:

$p(z) = z^4 + 2abz^2 - 2a(a + b)z^2 - 2b(a + b)z^2 + 4ab(a + b)z + d = z^4 +2(ab - a(a + b) - b(a + b))z^2 + 4ab(a + b)z + d =z^4 +2 (ab - a^2 - ab - ab - b^2)z^2 + 4ab(a + b)z + d = z^4 +2 [ab - (a^2 + 2ab + b^2)]z^2 + 4ab(a + b)z + d = z^4 + 2[ab - (a + b)^2]z^2 + 4ab(a + b)z + d$

This gives the soon well known iteration formula

$z \to z^4 + 2[ab - (a + b)^2]z^2 + 4ab(a + b)z + d$

critical points $z = a$, $z = b$, and $z = - (a + b)$. The set M_1 are those parameters (a, b, d) for which $z = a$ has a bounded orbit, the set M_2 are those parameters (a, b, d) for which $z = b$ has a bounded orbit, and the set M_3 are those parameters (a, b, d) for which $z = - (a + b)$ has a bounded orbit. The full quartic analogy to the Mandelbrot set are those parameters (a, b, d) for which all three critical points have bounded orbits, that is the set which is common to M_1, M_2, and M_3. As we have three complex parameters, ”a”, ”b”, and ”d” the quartic parameter space is a six dimensional hyper space. The special slice of the full quartic analogy when both ”a” and ”b” is fixed to zero, that is we iterate $z \to z^4 + d$, is the well known Quartic Mandelbrot set (see figure 3, the Generalized Mandelbrot set for Quartics).

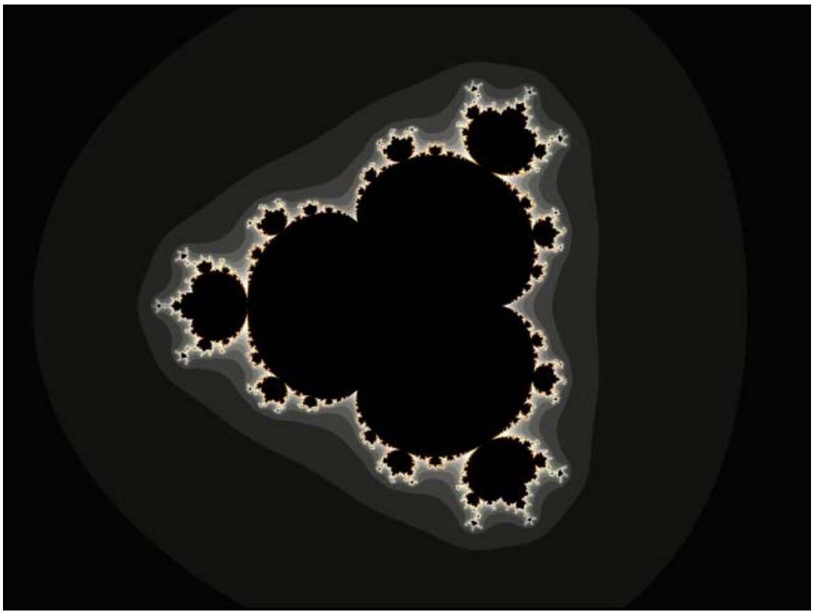

Fig 3. The Generalized Mandelbrot set for Quartics.

Many thanks to Stig for doing a hard theoretical work before we get the genial idea of how to set up the parameters from the great mathematicians, and for his wonderful submodule ”Quartic Parameterspace3”. It is included in his module sp3 and can be downloaded from my index page and soon from the data base of UF:

Now I leave to the diligent reader to continue to derive the optimal iteration formulas for d = 5 and even higher.

Regards

Ingvar