30) Short Introduction to Quartics

After quadratics, relating to the ordinary Mandelbrot set, comes cubics, and after cubics comes naturally quartics. The term ”quartics” refers to fourth-degree polynomials, which means the highest power of the variable ”z” is four. According to Article 28 in this chaotic series the smartest way to parameterize quartics, viewed as a dynamical system is:

$$z \to z^4 + 2[ab - (a + b)^2]z^2 + 4ab(a + b)z + c$$

(since the technical procedure for deriving the quartic iteration-formula is done in Article 28, we write the last parameter as ”c” instead of ”d”) with the three critical points $z_1 = a$, $z_2 = b$, and $z_3 = -(a + b)$. As we have three complex parameters, a, b, and c, the full quartic parameter space is a six-dimensional hyper-space. Accomplishing this article there is a 2D-slice drawn in the following way. The plane we have plotted is ”a_{real}, a_{imag}” after fixing the parameter b to 0.3+0i, and the parameter c to 0+0i. That means we have moved 0.3 units along the axis a_{real}. In the figure 1 ”M_1” the set M_1 is plotted, the set for which the first critical point $z_1 = a$ has a bounded orbit. In figure 2 ”M_2” the set M_2 is plotted, the set for which the second critical point $z_2 = b$ has a bounded orbit. Finally in figure 3 ”M_3” the set M_3 is plotted, the set for which the third critical point $z_3 = -(a + b)$ has a bounded orbit. Now let's use the layering technique in UF in a very technical way. We draw each subset in a separate layer in order to obtain a complete view of this 2D-slice of the six-dimensional fractal monster. This is done in figure 4 ”QuarticDemo”. This image has exactly the same coordinates as the three earlier images. The fact that the image is tilted to the left indicates that the image, although symmetric, is not centered at the center of the a-plane. The illustration shows that both the subsets M_1 and M_3 is entirely situated inside M_2 in this special slice. The black area represents QCL, the quartic connectedness locus, the set that is common to all the three subsets That is the set for which all three critical points have bounded orbits, the full quartic analogy for the Mandelbrot set for quadratics. The yellow rectangle shows where figure 5 ”QuarticDemo-mag” is zoomed in.

In this image, ”QuarticDemo-mag”, four arrows are drawn denoting the spots from which the a-parameter-seeds are obtained which together with the fixed parameters b = 0.3+0i, and c = 0+0i give rise to the four quartic Julia sets QJ1, QJ2, QJ3, and QJ4 (figures 6 - 9). Now let's turn over to the dynamical plane, that is the z-plane, and have a look at these four Julia sets one after another. In all four images green arrows appoint the three critical points. Note that ”b” always is 0.3+0i, while the value of ”a” and also ”- (a + b)” = ”- (a + 0.3)” depends on the spot selected from the plotted a-plane (you can see these values in the parameter-files).

1) QJ1: Non of the three critical points belongs to the filled-in Julia set (the ”filled-in Julia set” is the Julia set plus eventually enclosed region, see Article 1). That is all three critical points tend to infinity under iteration. Thus the Julia set is a Cantor set, that is an infinite hierarchy of dust. The parameter-seed for this Julia set belongs to none of the three sets M_1, M_2, and M_3.

2) QJ2: In this case the critical point z = b belongs to the filled-in Julia

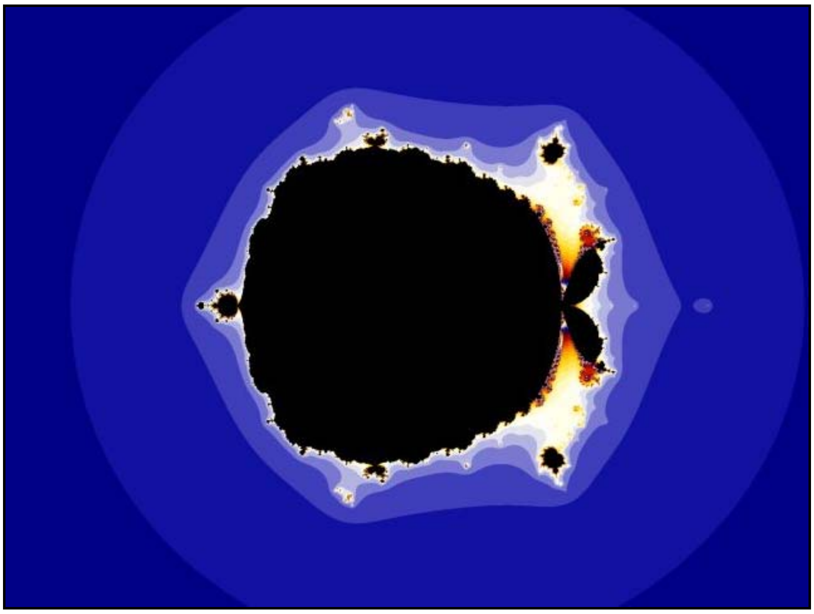

Fig 1. M_1.

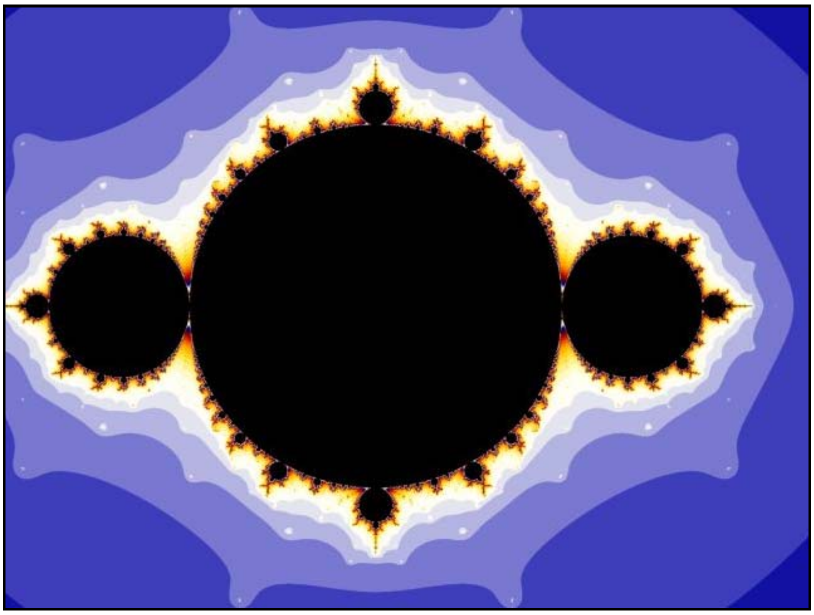

Fig 2. M_2.

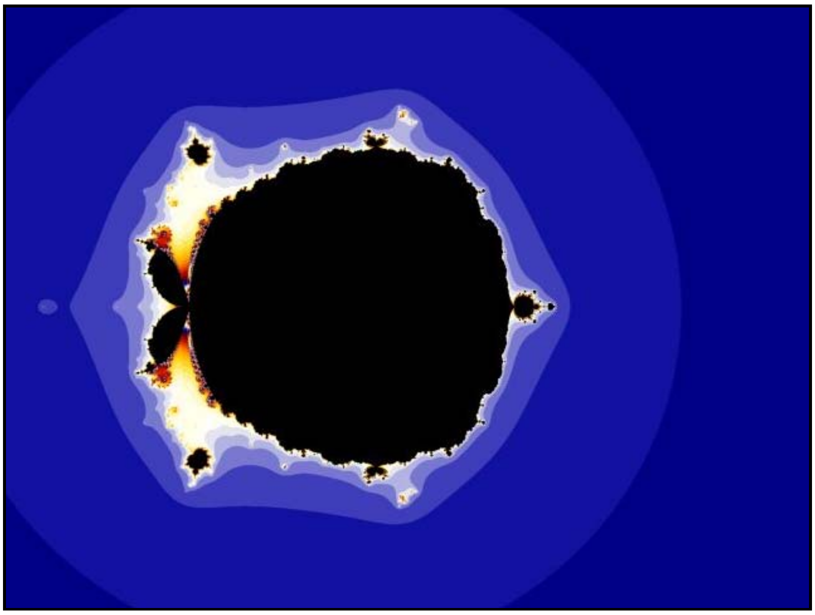

Fig 3. M_3.

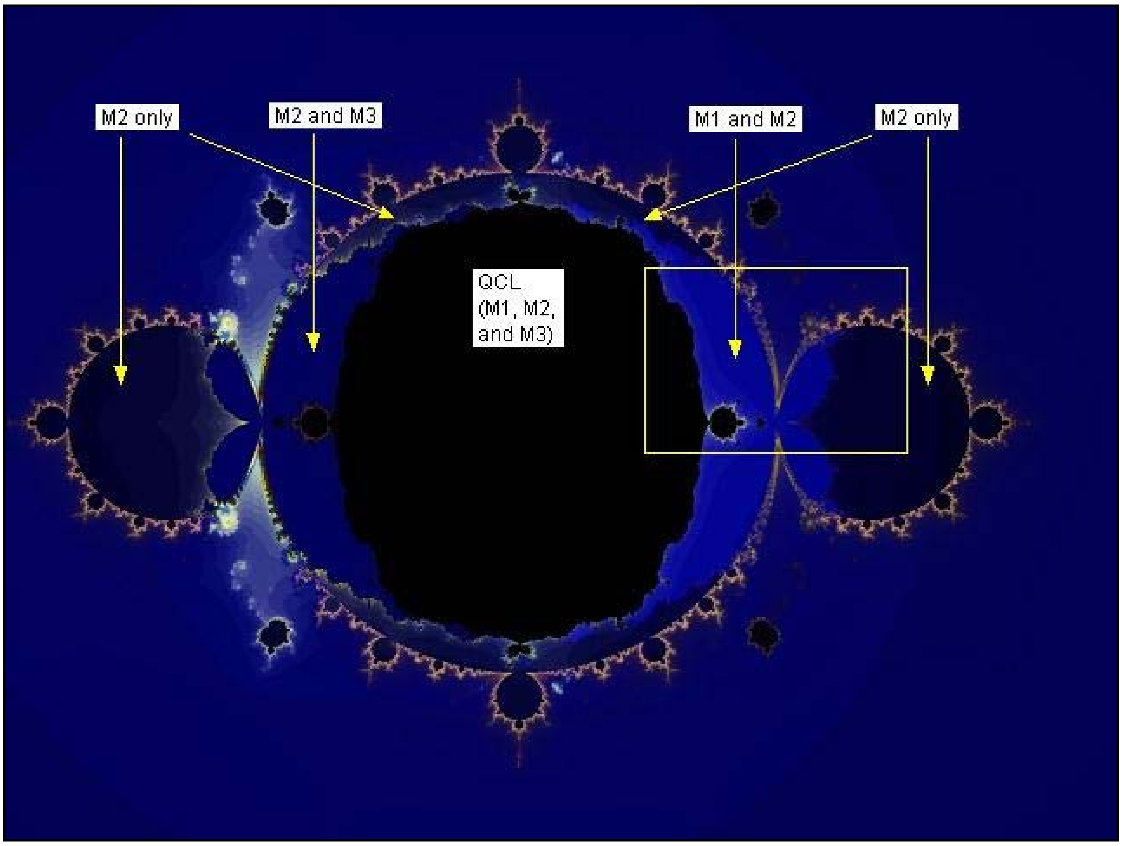

Fig 4. All sets.

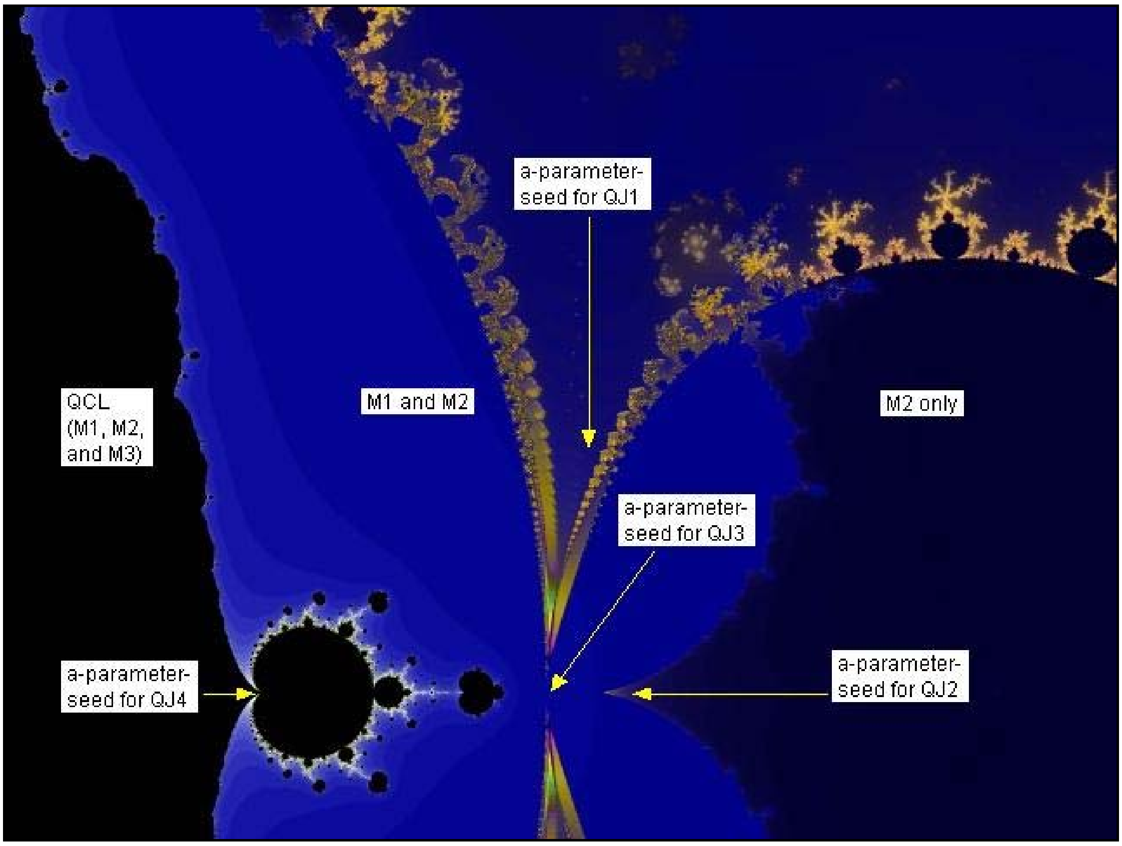

Fig 5. All sets, detail.

set and thus has a bounded orbit. The parameter-seed for this Julia set belongs to M_2, but not M_1 or M_3. The two other critical points tend to infinity under iteration. Also this Julia set is disconnected, but enclosed regions are there.

3) QJ3: In this case, besides z = b, also the critical point z = a belongs to the filled-in Julia set and thus has a bounded orbit. The parameter-seed for this Julia set belongs to both M_1 and M_2, but not M_3. Like in the previous case the Julia set is disconnected and enclosed regions are also there. However these enclosed regions have been larger. In fact clusters of enclosed regions are "half" as many in number compared with the previous case. I said "half" because it's a "half infinity".

4) QJ4: Here all the three critical points belong to the filled-in Julia set, and the resulting Julia set is connected and disc-like. The parameter-seed belongs to all three subsets in the parameter space, M_1, M_2, and M_3. That is because this Julia set is connected we say that this parameter-seed belongs to "quartic connectedness locus".

I strongly recommend the diligent reader to play around with Switch mode (see Article 6) in ”QuarticDemo” and ”QuarticDemo-mag”. Moving the cursor around and in the different subsets you will clearly see the above described phenomena regarding the resulting Julia sets.

In a six-dimensional hyper-space there are 15 systems of perpendicular 2D-slices. Due to certain symmetries in this setup of quartic parameter space some are equal. For example the only difference between ”b_{real}, b_{imag}” and ”a_{real}, a_{imag}” is that M_1 and M_2 have altered. The resulting Julia sets are exactly the same. That’s the reason why the old module ”Quartic Parameterspace” only

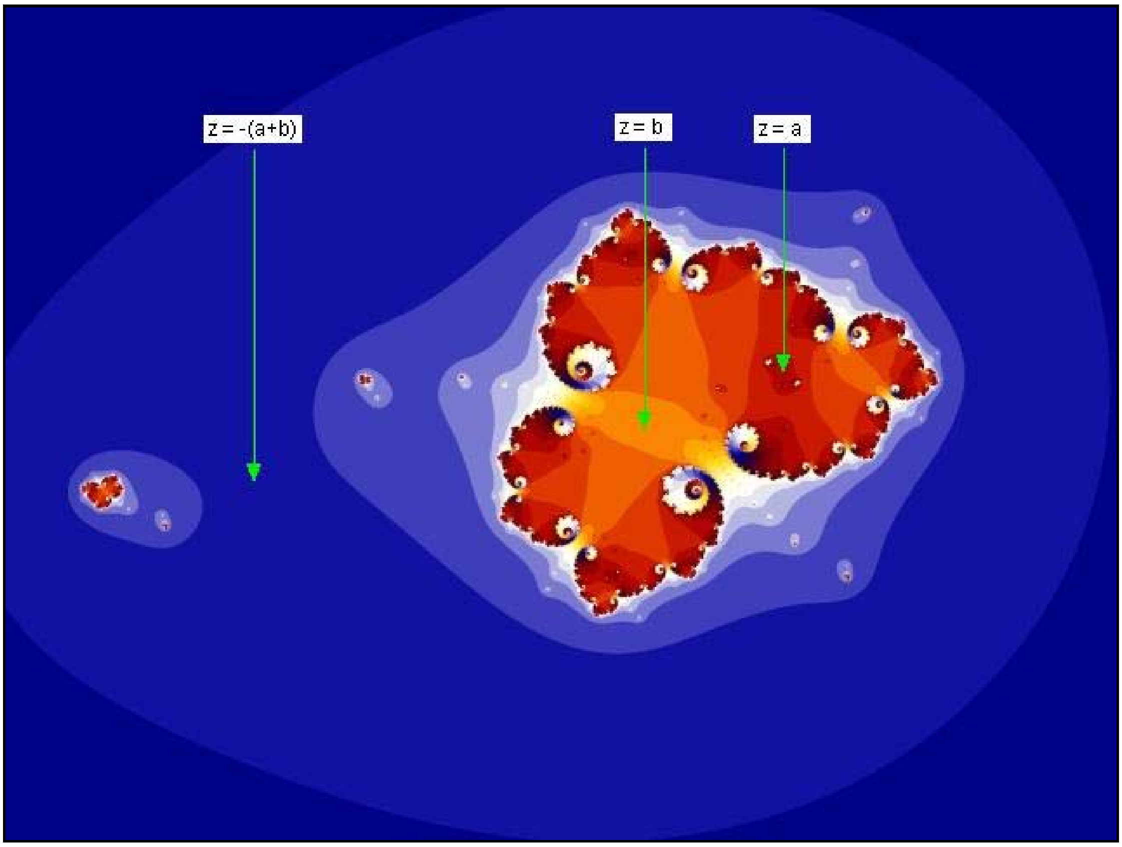

Fig 6. QJ1.

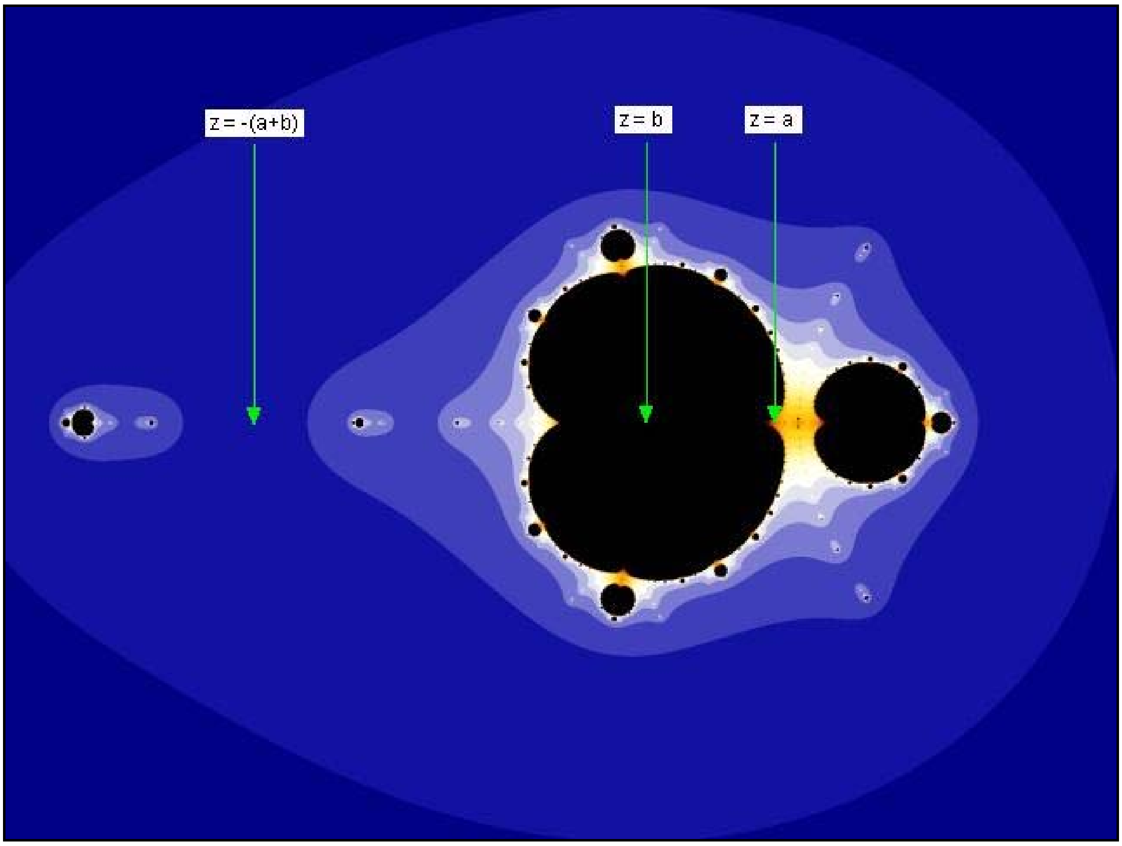

Fig 7. QJ2.

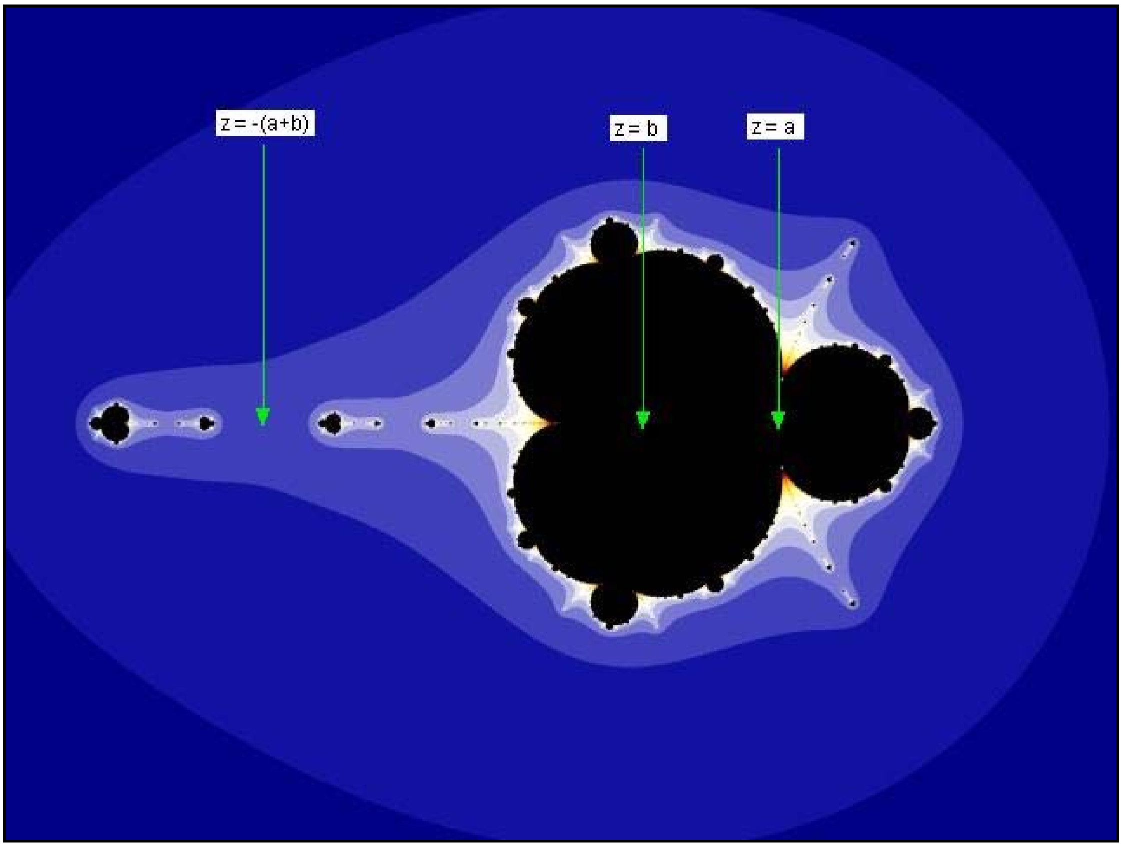

Fig 8. QJ3.

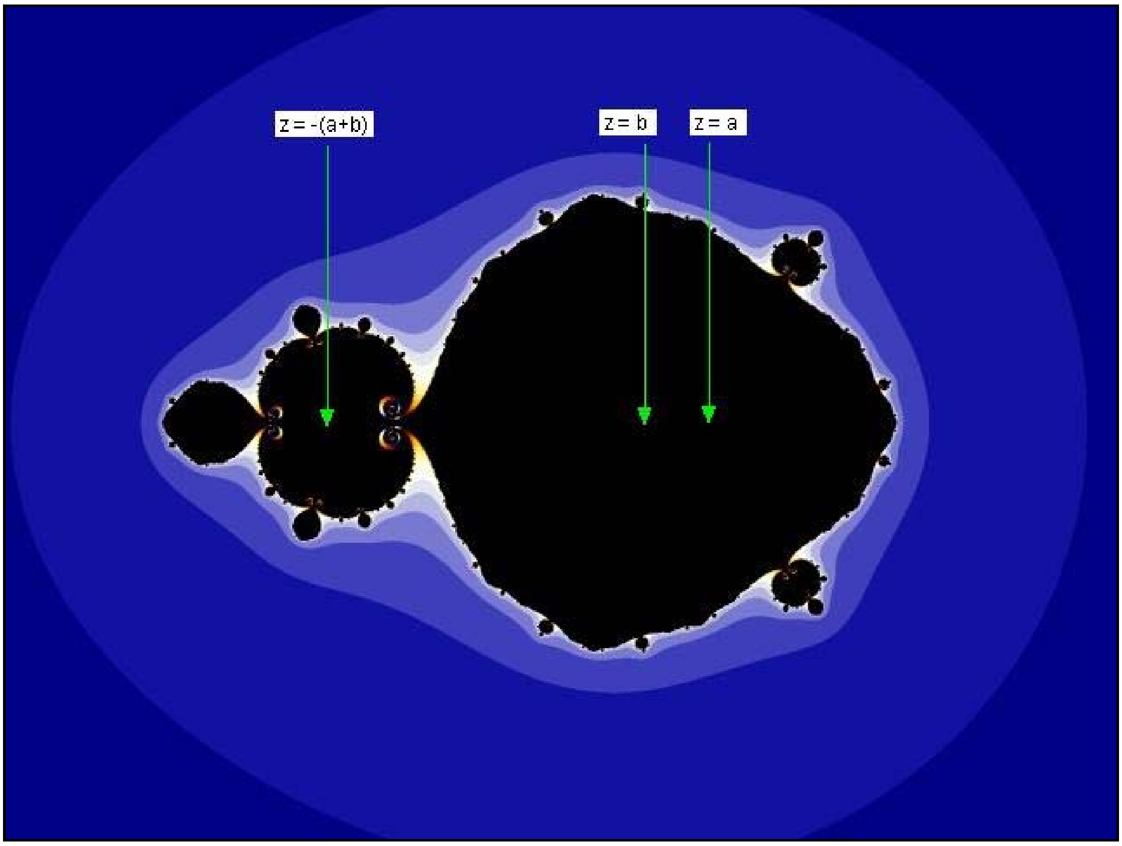

Fig 9. QJ4.

maintains 9 systems. However in the new one ”Quartic Parameterspace 3”, for pedagogical reaseans, all 15 systems of planes are there.

Regards

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