31) More about Quartics

(Before reading this doubble article, I strongly recommend a repetition of Article18)

In this article we shall have a little more look about some features of quartics. In most slices the features are almost the same as in slices of cubics not intersecting zero, see Article18. In some slices two of the three subsets coalesces and in two of these cases the non-coalescing subset have an infinite range. So is the case in (a_{real}, a_{imag}) intersecting zero where M_1 and M_3 coalesce and the feature reminds about ”CCAP-cubics” (see my ik-module) and M_2 is stretched to infinity. Exact the same phenomena occur in (b_{real}, b_{imag}) as this system of slices is identical with (a_{real}, a_{imag}) despite from that M_1 and M_2 have changed place. Due to this symmetry this slice-system is not there in the old quartic module of Stig.

Now I will tell a little story. In the middle of the nineties I investigated slices of parameter spaces for polynomials with software available for me at that time (a plug-in to MandelZot, MacOS, by Dave Platt). My way to cubics during that time can be read if you click ”Pictures from the Cubic Parameterspace” at my index-page. Further more. If I iterated the quartic formula $z \rightarrow z^4 + kz^2 + c$, so that $z = 0$ was one of the critical points (the above mentioned software always used $z = 0$ as input when drawing a parameter-plane), then there were copies of the ordinary quadratic Mandelbrot set in the same way as in cubics if ”k” was non-zero. On the other hand when iterating $z \rightarrow z^4 + kz^3 + c$, $z = 0$ still one critical point (in fact two), there were copies of the generalized cubic Mandelbrot set ($z \rightarrow z^3 + c$)! Figure 1, ”$z^4 + z^3 + c$” shows an example drawn with MandelZot. To the "acupuncture-points" of this cubic minibrot there are components having the shape of connected cubic Julia sets (compare the phenomena described in Article 18). This indicates that the black set is situated inside a subset, showing that the black components are parts of quartic connectedness locus (qcl). This was also

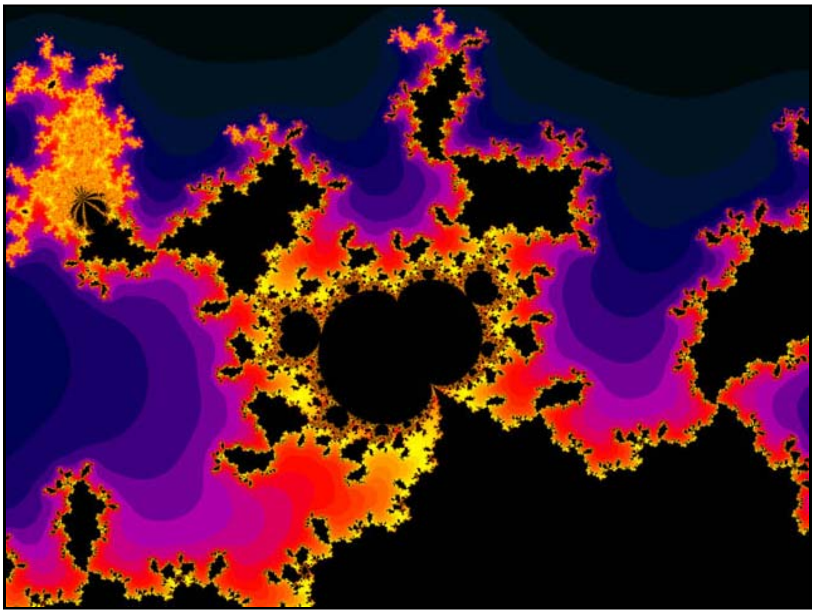

Fig 1. $z \rightarrow z^4 + z^3 + c$, detail.

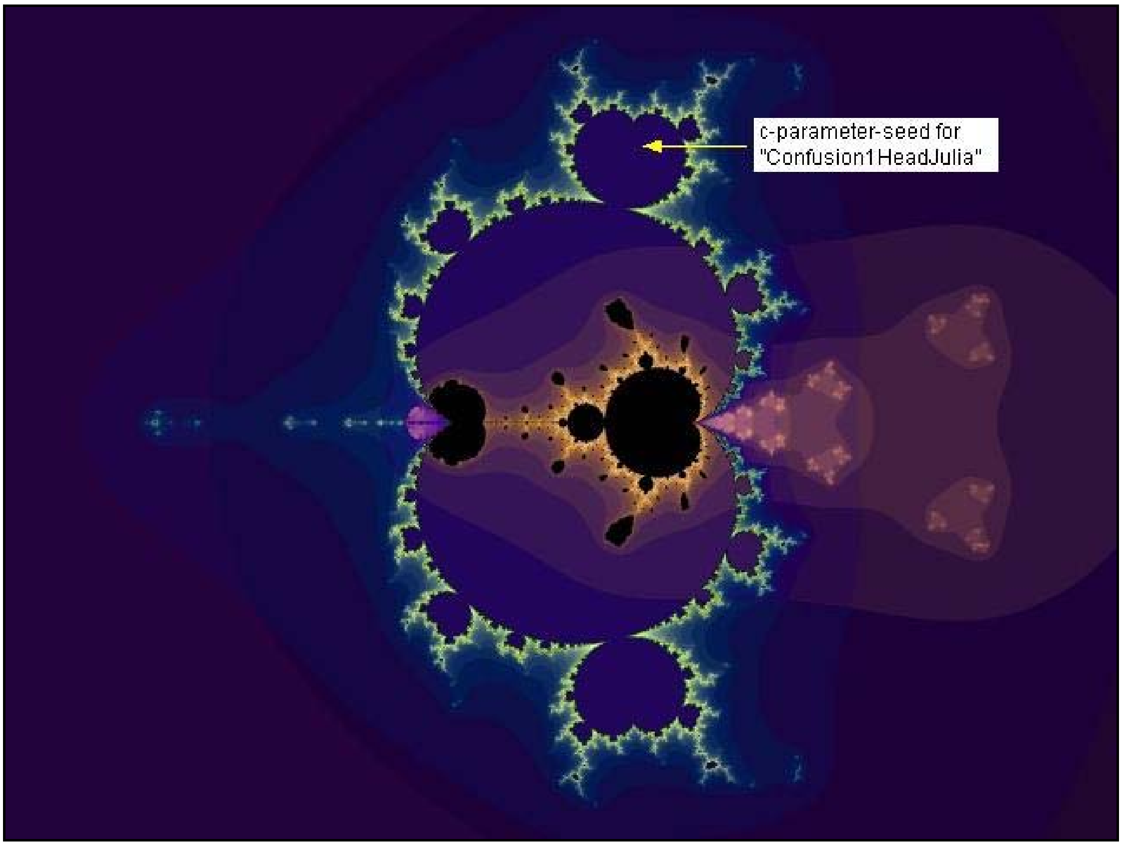

Fig 2. Confusion1.

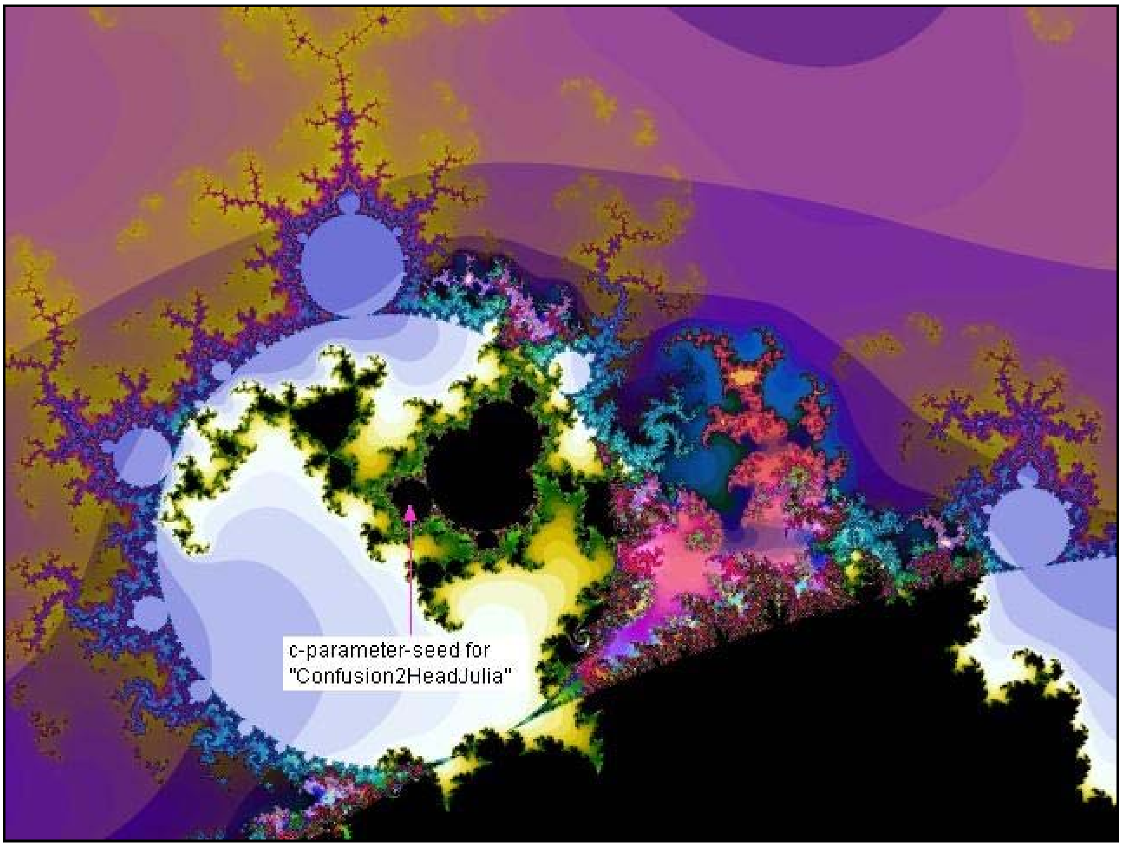

Fig 3. Confusion2.

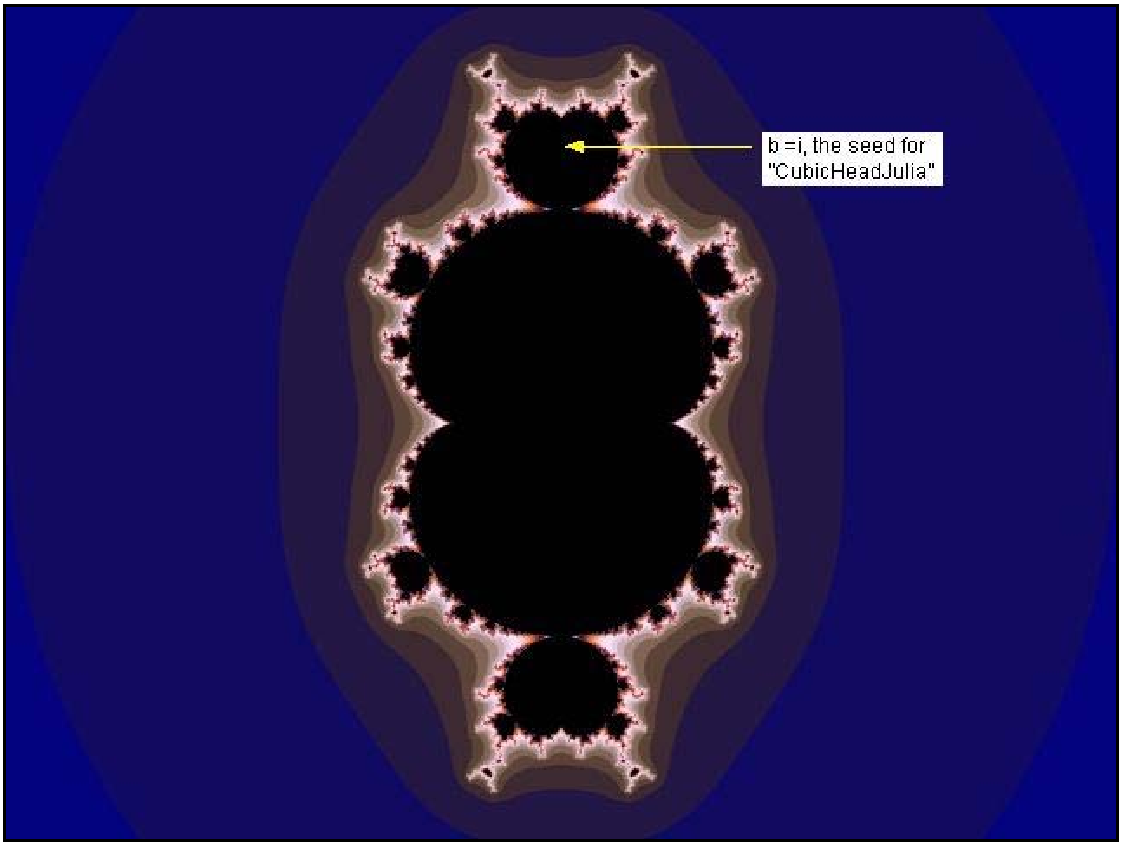

Fig 4. Cubic Mandel.

verified in as much as parameter-seeds from this set gave rise to connected Julia sets.

Now in a full-parameterized quartic iteration formula generalized cubic Mandelbrot sets also ought to be found in some slices. Thus was a statement by me, but wasn't found in the earlier quartic-formulas written by Stig. However that happy day in the earlier summer 2001 when the optimal iteration-formula was found by me and Stig, inspired by a magic page in a paper (referred to in Article 28) by professor Bodil Branner, this phenomena was discovered first by Stig then by me in the evening the same day. The optimal iteration formula runs:

$z \rightarrow z^4 + 2[ab - (a + b)^2]z^2 + 4ab(a + b)z + c$ with
the critical points $z_1 = a$, $z_2 = b$, and $z_3 = -(a + b)$.

In this formula slices containing copies of the generalized cubic Mandelbrot set can be found in c-slices if ”a” equal to ”b” not equal to zero. If $a = b = 0$ you actually iterate $z \rightarrow z^4 + c$ and receive the generalized quartic Mandelbrot set on the c-plane. On the other hand if $a = b$ not equal to zero two of the three subsets coalesce and in the border of these common subsets there are copies of the cubic Mandelbrot set. Some of these copies have their border against the escape-locus, the locus where all three critical orbits escape, see figure 2, ”Confusion1”. The others are inside the non-coalescing set. These later cubic minibrots have cubic Julia-like components attached to it, as in figure 3, ”Confusion2”. However there was a big surprise for me. I've expected the border of the non-coalescing set against escape-locus to have the shape of the cubic Mandelbrot set. However this subset turns out to have the shape of

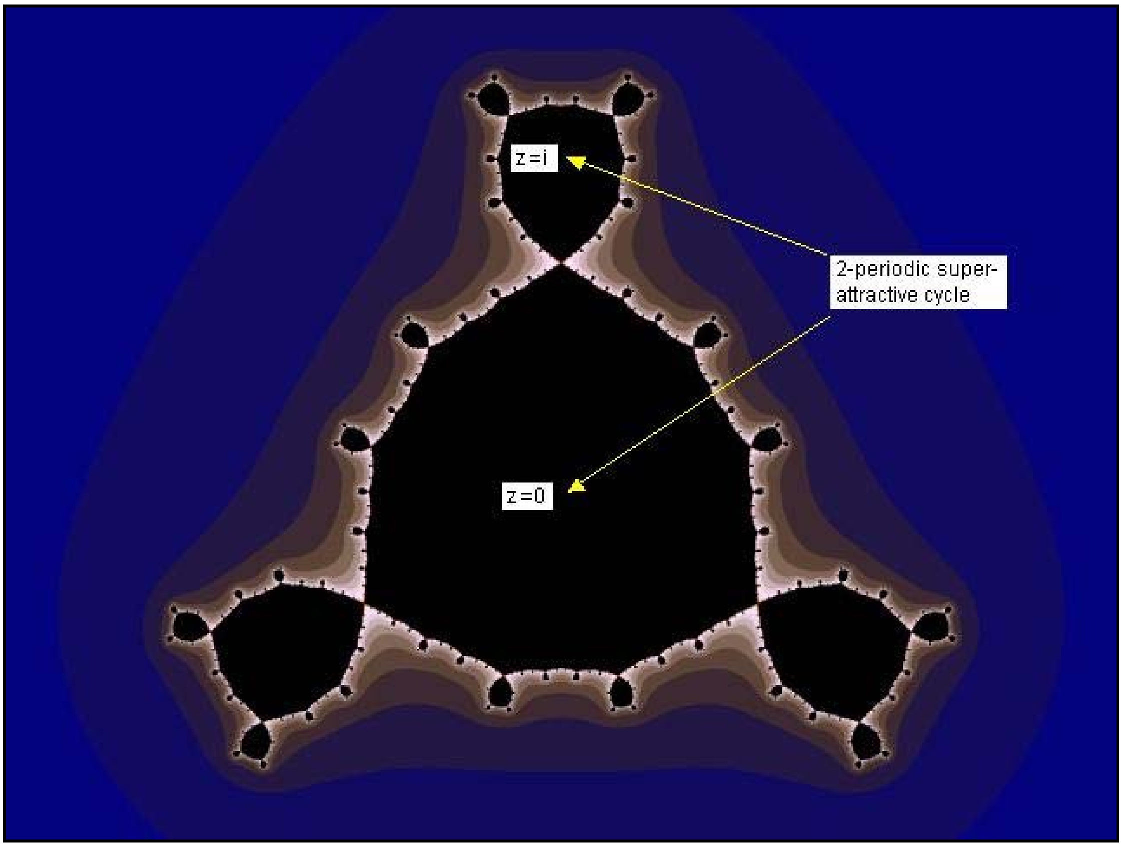

Fig 5. Cubic HeadJulia.

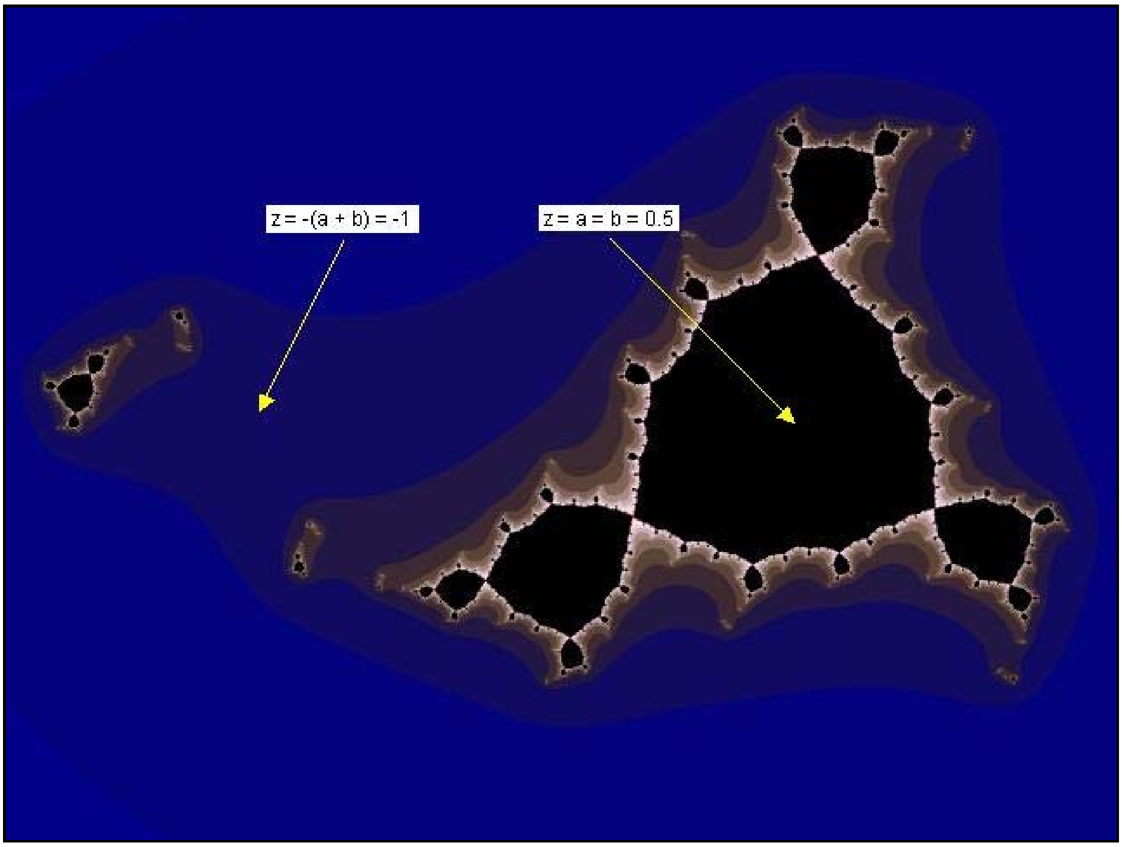

Fig 6. Confusion1 HeadJulia.

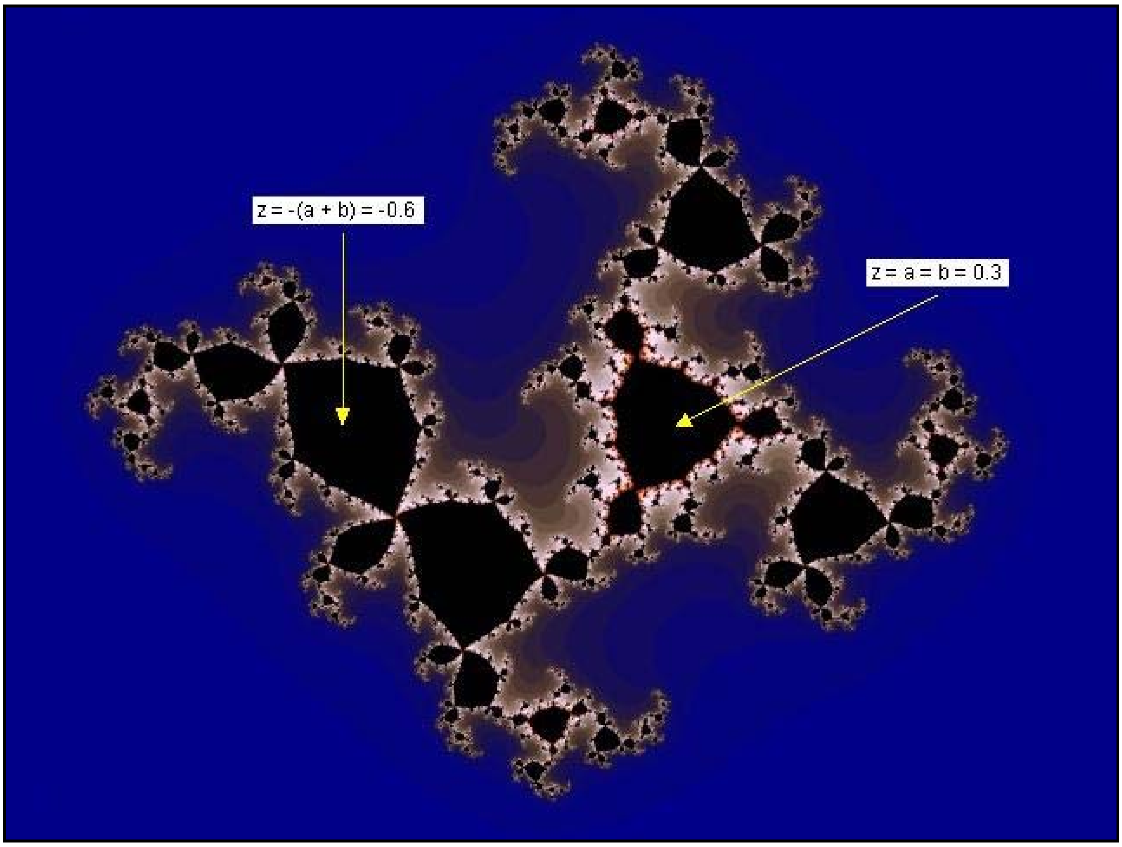

Fig 7. Confusion2 HeadJulia.

the ordinary quadratic Mandelbrot set (see ”Confusion2”). In a like manner when parts of the cubic Mandelbrot set has its border against escape-locus, the border of qcl inside those parts of the cubic Mandelbrot set have components of the quadratic Mandelbrot set with quadratic Julia-like components attached to its acupuncture-points (as in ”Confusion1”).

Now let's have a look at the consequences for the dynamical plane where the iterations takes place and the Julia sets reside. First let's have a look at the prototype cubic Mandelbrot set figure 4, "CubicMandel". From the ”center” of the upper head, b = i, we pick up the parameter-seed and draw a Julia set for this seed figure 5, ”CubicHeadJulia”. This Julia set is accomplished by a 2-periodic super-attractive cycle, that is 0 -> i -> 0 -> i -> 0 -> i etc (i^3 + i = i*i*i + i = -1*i + i = -i + i = 0). Note that the cubic Mandelbrot set is a special case (a special 2D-slice of the four-dimensional cubic connectedness locus) of the iteration-formula $z \to z^3 - 3a^2 z + b$ where a = 0 and the critical points, $z_1 = +a$ and $z_2 = -a$ coalesces in zero. Figures 6 - 7 ”Confusion1-HeadJulia” and ”Confusion2- HeadJulia” are the results from the appointed parameter-seeds taken from one of the heads in ”Confusion1” and ”Confusion2”, the full six-dimensional coordinates for these parameter-seeds being respectively:

$*a_{real} = 0.5$
$*a_{imag} = 0$
$*b_{real} = 0.5$
$*b_{imag} = 0$

*c_{real} = 0.4125 (plotted horizontal coordinate)
*c_{imag} = 0.6833333334 (plotted vertical coordinate)

giving rise to the Julia set ”Confusion1HeadJulia”:

$z \to z^4 + 2[ab - (a + b)^2]z^2 + 4ab(a + b)z + c =$
$z^4 + 2[0.5*0.5 - (0.5 + 0.5)^2]z^2 + 4*0.5*0.5 (0.5 +$
$0.5)z + (0.4125 + 0.6833333334 i) =$
$z^4 - 1.5z^2 + z + (0.4125 + 0.6833333334i)$

and

*a_{real} = 0.3
*a_{imag} = 0
*b_{real} = 0.3
*b_{imag} = 0
*c_{real} = -0.560081525106 (plotted horizontal coordinate)
*c_{imag} = 0.28144217424 (plotted vertical coordinate)

giving rise to the Julia set ”Confusion2HeadJulia”:

$z \to z^4 + 2(ab - (a + b)^2)z^2 + 4ab(a + b)z + c =$
$z^4 + 2(0.3*0.3 - (0.3 + 0.3)^2)z^2 + 4*0.3*0.3 (0.3 +$
$0.3)z + (-0.560081525106 + 0.28144217424i) =$
$z^4 + 2(0.09 - 0.36)z^2 + 0.36*0.6z + (-0.560081525106$
$+ 0.28144217424i) = z^4 - 0.54z^2 + 0.216z + (-0.560081525106 +$
$0.28144217424i)$

In the first case, ”Confusion1HeadJulia”, two critical points have bounded orbits and are situated inside the filled-in Julia set, and the third critical point, outside the filled-in Julia set, escapes to infinity. Oh yeah, there are only two arrows, but the first two critical points coalesces since $z_1 = a = z_2 = b = 0.5$. In fact the enclosed regions have the same shape as ”CubicHeadJulia” where also two critical points coalesce.

In the second case, ”Confusion2HeadJulia” all three critical points belongs two the filled-in Julia set and have bounded orbits. Since all critical points have bounded orbits this Julia set is connected and disc-like. As in the first case two critical points coalesce, since $z_1 = a = z_2 = b = 0.3$. This double critical point is also situated in a region having the shape of ”CubicHeadJulia”.

Due to the lenght and many files of this article, I've splitted it to two parts. The second part will be posted probably already within 10 hours. So long :-)

Regards

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